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**LETTER TO THE EDITOR**

# Scattering of plane waves in self-dual Yang–Mills theory

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**Abstract.** We consider the classical self-dual Yang–Mills equation in $(3 + 1)$-dimensional Minkowski space. We have found a new solution. It describes the scattering of $n$ plane waves. The construction which we use is similar to the quantum inverse scattering method. We introduce a 'Monodromy matrix' $\hat{T}$. It acts in the direct product of the universal enveloping of $SU(N)$ algebra and an auxiliary linear space. In order to obtain the solution of the self-dual Yang–Mills equation, we take a special matrix element of $(1 - \hat{T})^{-1}$ in the auxiliary space.

We consider a classical Yang–Mills field valued in the $SU(N)$ algebra, defined over $(3+1)$-dimensional Minkowski space. We study the self-dual equation:

$$F_{\mu\nu} = \frac{\mathrm{i}}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{1}$$

The study of this self-dual Yang–Mills equation is important for the understanding of QCD [1–7].

Following [8], we take the light-cone gauge $A_{0-z} = 0$. Then the self-dual Yang–Mills equation leads to the relations

$$A_{x+\mathrm{i}y} = 0 \qquad A_{0+z} = \sqrt{2}\partial_{x+\mathrm{i}y}\Phi \qquad A_{x-\mathrm{i}y} = \sqrt{2}\partial_{0-z}\Phi. \tag{2}$$

Here $A_{0\pm z} = A_0 \pm A_z$, $A_{x\pm \mathrm{i}y} = A_x \mp \mathrm{i}A_y$[+] and $\Phi$ is a scalar $SU(N)$-valued field which satisfies the following equation:

$$\Box\Phi - \mathrm{i}g[\partial_{x+\mathrm{i}y}\Phi, \partial_{0-z}\Phi] = 0. \tag{3}$$

This is associated with a cubic action [9]. Following [10], we start looking for the solution of equation (3) using perturbation theory in the coupling constant $g$:

$$\Phi(x) = \sum_{m=1}^{\infty}\Phi^{(m)}(x). \tag{4}$$

Here, $\Phi^{(m)}$ depends on the coupling as $g^{m-1}$. The first term satisfies a linear equation

$$\Box\Phi^{(1)} = 0. \tag{5}$$

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+ We use the coordinate $X^{0\pm z} = (t \pm z)/2$ and $X^{x\pm \mathrm{i}y} = (x \pm \mathrm{i}y)/2$ with metric $g_{0+z,0-z} = -g_{x+\mathrm{i}y,x-\mathrm{i}y} = 2$.

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We choose $\Phi^{(1)}$ as a sum of $n$ plane waves

$$\Phi^{(1)}(x) = -\mathrm{i}\sum_{j=1}^{n} T^{a_j}\,\mathrm{e}^{-\mathrm{i}k_j x} f(k_j). \tag{6}$$

Here $T^a$ are $SU(N)$ generators

$$[T^a, T^b] = \mathrm{i}\sqrt{2} f^{abc} T^c \qquad \mathrm{tr}\, T^a T^b = \delta^{ab}. \tag{7}$$

The index $a_j$ specifies a 'colour' of the $j$th plane wave. The $k_j$ are a set of $n$ different light-cone vectors $k_j^2 = 0$, and $f(k)$ is a function with support on the light-cone. We will also use the following notation:

$$Q_j = \frac{(k_j)_{0+z}}{(k_j)_{x+\mathrm{i}y}} = \frac{(k_j)_{x-\mathrm{i}y}}{(k_j)_{0-z}}. \tag{8}$$

We have found explicit expressions for the $\Phi^{(m)}$. The first two terms coincide with the results of [10], but all other $\Phi^{(m)}$ $(m \geqslant 3)$ are different.

Let us explain our solution. We shall use an abbreviation:

$$\phi(j) = T^{a_j}\,\mathrm{e}^{-\mathrm{i}k_j x} f(k_j). \tag{9}$$

We introduce the following function:

$$V(a) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} a^n = \oint \frac{\mathrm{d}t}{2\pi\,\mathrm{i}} \frac{\mathrm{e}^{1/t+at}}{t} = I_0(2\sqrt{a}). \tag{10}$$

Here $I_0$ is a modified Bessel function of the first kind. The integration contour is a circle around zero. We integrate in the positive direction.

Let us define a linear operator $\hat{T}$ by giving its kernel:

$$T(\alpha_1, \alpha_2; j_1, j_2) = g\phi(j_1) P(j_1, j_2) \times \int_0^\infty \mathrm{d}s\ \ \mathrm{e}^{-s} V(s\alpha_1 g\phi(j_1) P(j_1, j_2)) V(s\alpha_2 g\phi(j_2) P(j_1, j_2)). \tag{11}$$

Here $j_1$ and $j_2$ run through $n$ values. The integration variables $\alpha_{1,2}$ take values in the unit interval $[0, 1]$. We shall consider $T$ as an operator acting on a direct product of $n$-dimensional vector space and the space of functions on the unit interval. The kernel $T(\alpha_1, \alpha_2; j_1, j_2)$ takes its values in the universal enveloping algebra of $SU(N)$. We are using $P(j_1, j_2)$ which is defined by

$$P(j_1, j_2) = \begin{cases} (Q_{j_1} - Q_{j_2})^{-1} & \text{for } j_1 \neq j_2 \\ 0 & \text{for } j_1 = j_2. \end{cases} \tag{12}$$

The kernel $T(\alpha, \alpha'; j, j')$ depends only on the $j$th and $j'$th plane waves. It vanishes if $j = j'$.

The function (11) is a kernel of an operator $\hat{T}$

$$(\hat{T})_{(\alpha_1; j_1), (\alpha_2; j_2)} = T(\alpha_1, \alpha_2; j_1, j_2)$$

whose index $(\alpha; j)$ takes values in $[0, 1] \times \{1, 2, \ldots, n\}$. It acts on a 'vector' $(\boldsymbol{f})_{(\alpha; j)}$ (which takes its value in the universal enveloping algebra) as follows:

$$(\hat{T}\boldsymbol{f})_{(\alpha; j)} = \sum_{j'=1}^{n} \int_0^1 \mathrm{d}\alpha'\, T(\alpha, \alpha'; j, j') (\boldsymbol{f})_{(\alpha'; j')}. \tag{13}$$

$\hat{T}$ can be compared with the monodromy matrix of the quantum inverse scattering method. It acts on the direct product of the universal enveloping algebra of $SU(N)$ and an auxiliary space. The auxiliary space is also a direct product of $n$-dimensional vector space and a linear space of functions defined on the unit interval $0 \leqslant \alpha \leqslant 1$. We call $\hat{T}$ the 'monodromy matrix'.

We introduce two special 'vectors' (see (9))

$$(\phi)_{(\alpha;j)} = \phi(j) \qquad (\phi_0)_{(\alpha;j)} = 1. \tag{14}$$

For example, a scalar product of $\phi_0$ and an arbitrary vector function $\boldsymbol{f}$ is equal to

$$\phi_0 \cdot \boldsymbol{f} = \sum_{j=1}^{n} \int_0^1 \mathrm{d}\alpha\, (f)_{(\alpha;j)}.$$

Now all the notation is prepared to allow us to write down the solution of the self-dual equation (3) that we have found:

$$\Phi(x) = -\mathrm{i}\phi_0 \cdot \left( \frac{1}{1-\hat{T}} \right) \phi. \tag{15}$$

This is the main result of our paper.

The operator $(1-\hat{T})^{-1}$ in equation (15) is defined by the infinite series

$$\Phi(x) = -\mathrm{i}\phi_0 \cdot \left( \sum_{l=0}^{\infty} (\hat{T})^l \right) \phi. \tag{16}$$

The exact expression for each term

$$\tilde{\Phi}^{(l)}(x) = -\mathrm{i}\phi_0 \cdot (\hat{T})^{l-1} \phi \tag{17}$$

is given by

$$\tilde{\Phi}^{(l)}(x) = -\mathrm{i} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \ldots \sum_{j_l=1}^{n} \int_0^1 \mathrm{d}\alpha_1 \int_0^1 \mathrm{d}\alpha_2 \ldots$$
$$\ldots \int_0^1 \mathrm{d}\alpha_l\, T(\alpha_1, \alpha_2; j_1, j_2) T(\alpha_2, \alpha_3; j_2, j_3) \ldots T(\alpha_{l-1}, \alpha_l; j_{l-1}, j_l) \phi(j_l). \tag{18}$$

The proof of formulae (15) and (16) can be given as follows. One decomposes the self-dual equation (3) into a Taylor series in the coupling constant $g$. Then one explicitly evaluates each term. One must make sure that this perturbative series satisfies the self-dual equation (3). All the details of the calculations can be found in [11].

*Remark 1.* We can perform the $s$-integration in the definition of the 'monodromy matrix' (11) using the formula for the Bessel function. If $\phi(j_1)$ and $\phi(j_2)$ in equation (11) commute, then the result can be written in terms of the exponential function and $V(a)$ (10) (or the modified Bessel function $I_0$).

*Remark 2.* Our formulae are complicated. So let us study them in a simplified situation. Let us consider what will happen to our formulae if all $n$ generators $T^{a_j}$ belong to a Cartan subalgebra of $SU(N)$ algebra:

$$[\phi(j), \phi(j')] = 0 \qquad \forall j,\ j' = 1, \ldots, n \tag{19}$$

in (9). For this case, a cancellation between many terms gives a trivial solution:

$$\Phi^{(l)}(x) = 0 \qquad \text{for } l \geqslant 2. \tag{20}$$

The result is the sum of $n$ place waves which we chose as the input of the iteration:

$$\Phi(x) = \Phi^{(1)}(x) = -\mathrm{i}\phi_0 \cdot \phi = -\mathrm{i} \sum_{j=1}^{n} T^{a_j}\, \mathrm{e}^{-\mathrm{i}k_j x} f(k_j). \tag{21}$$

*Remark 3.* To derive our formula (15), we have not used any properties of the $SU(N)$ generators. So our formula is valid not only for $SU(N)$ but also for other gauge groups. The two requirements to obtain the formula is that $\phi(j)$ in (9) satisfies the free equation and all momenta $k_j$ $(j = 1, \ldots, n)$ are different in order that $P(j_1, j_2)$ in (12) are well defined. We can choose $\phi(j)$ as some linear combination of the generators of the gauge group:

$$\phi(j) = -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}k_j x}\left(\sum_{a_j} f_{a_j}^{(j)}(k_j) T^{a_j}\right).$$

Here, $f_{a_j}^{(j)}(k_j)$ is a function with support on the light-cone. This linear combination can be interpreted as a set of plane waves with various colours but with the same momentum $k_j$. So we can treat the case of a set of particles having the same momenta.

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